

## Exact black holes and gravitational shockwaves on codimension-2 branes

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**ABSTRACT:** We derive exact gravitational fields of a black hole and a relativistic particle stuck on a codimension-2 brane in  $D$  dimensions when gravity is ruled by the bulk  $D$ -dimensional Einstein-Hilbert action. The black hole is locally the higher-dimensional Schwarzschild solution, which is threaded by a tensional brane yielding a deficit angle and includes the first explicit example of a ‘small’ black hole on a tensional 3-brane. The shockwaves allow us to study the large distance limits of gravity on codimension-2 branes. In an infinite locally flat bulk, they extinguish as  $1/r^{D-4}$ , i.e. as  $1/r^2$  on a 3-brane in  $6D$ , manifestly displaying the full dimensionality of spacetime. We check that when we compactify the bulk, this special case correctly reduces to the  $4D$  Aichelburg-Sexl solution at large distances. Our examples show that gravity does not really obstruct having general matter stress-energy on codimension-2 branes, although its mathematical description may be more involved.

**KEYWORDS:** Large Extra Dimensions, Black Holes, p-branes.

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## 1. Introduction

The emergence of the braneworld paradigm has spurred a lot of work in the exploration of gravity in spaces with defects and/or boundaries of various codimension. Among the higher-codimension setups, the codimension-2 branes [1]–[6] gained attention because in asymptotically locally flat environs, their tension curves only the two transverse directions, cusping them into a cone centered at the location of the brane. This behavior is modified for different bulk asymptotics [4] and for branes residing on intersections of codimension-1 objects [7, 8]. The attempts to use this ‘off-loading’ of the brane vacuum energy into the bulk for alleviating the 4D cosmological constant problem [5, 9] have been found to require the usual finely tuned adjustments of parameters once compactification is enforced to produce 4D gravity at large distances. Indeed, to get an intrinsically flat brane one must have very particular boundary conditions in the bulk, which requires adjusting<sup>1</sup> the bulk sector in some way to maintain the brane’s flatness upon a change of matter sector parameters [4], [10]–[12].

Nevertheless the curiosity that tensional branes can remain intrinsically flat provoked the study of setups with codimension-2 branes. Surveying the dynamics with a generic stress-energy on a thin brane in an empty bulk, [13] asserted that there is an inconsistency. They claimed that bulk Einstein’s equations describing codimension-2 branes with  $\delta$ -function stress-energy allowed only pure tension  $\lambda$ , with  $T_{\mu\nu} = -\lambda g_{\mu\nu} \delta^{(2)}(\vec{y})$  in longitudinal directions, and with vanishing transverse components. Otherwise, noted [13], the solutions would have featured stronger, non-distributional singularities, that seemed either unacceptable or difficult to contend with. To handle these problems frameworks with

<sup>1</sup>Such adjustments are by necessity global, in spite of the ‘local guise’ as a change of the conical angle; the change extends to the end of the world in the bulk due to the peculiarities of ‘transverse’ 2 + 1  $D$  gravity.

higher-dimensional operators in the bulk [14], thickened, regulated branes [15]–[17], and combinations thereof were considered [18]. The common goal of these investigations was to somehow isolate and tame geometric singularities in order to match geometry and brane stress-energy.

These are all reasonable first-pass strategies, which however should be pursued carefully since such approaches could be dangerous, and even deceptive. Because gravity is a theory with a cutoff, its short distance limits are very tricky. Indeed, pathologies with distributional sources, similar to those encountered in codimension-2 setups [13]–[18] are already familiar in usual General Relativity (GR). Perhaps the simplest example arises from the Schwarzschild solution: in the linearized limit, one may be deceived to think of it as a field of a  $\delta$ -function source. In the full theory the short distance behavior is completely different from the linear theory. When the exterior geometry is followed inward, at short length scales the strong nonlinear gravity effects replace the apparent timelike singularity by a spacelike one, cloaking it with a horizon. Clearly, we do not throw away the Schwarzschild solution just because it does not have a  $\delta$ -function in its core. We cannot insist on retaining a  $\delta$ -function source because this source is itself an *approximation*, obtained by coarse-graining over the interior structure of a realistic lump of energy. At very short distances, this idealized form will be modified by corrections from interactions including gravity and also from quantum mechanics.

Many more examples are provided by line sources in GR [19]–[21]. It is well known that the singularities in that case are hard to even classify [19], and that the limiting procedures involving distributions that would reproduce the fields of static straight symmetric  $\delta$ -function sources are cumbersome and ambiguous [20]. Nevertheless, this has not hampered deriving cosmic string solutions and exploring their dynamics [21]. This program revealed that the thin strings in conical spaces with  $\delta$ -function stress-energy are really an idealization, and that in more realistic situations, when local strings wiggle or when they are perturbed by local inhomogeneities of matter on them, they will develop long range Newtonian potentials in transverse directions. Although this may modify the conical geometry at short and long ranges [21], as long as the asymptotic geometry very far from the disturbance relaxes to a conical space, they can be viewed as legitimate string configurations. An extreme case in point are the black holes pierced by cosmic strings found by Aryal, Ford and Vilenkin (AFV) [22], where the geometry of the local string asymptotes a line distribution with a conical deficit far away from the black hole, but is tremendously deformed near the hole by its strong nonlinear fields. While it was not immediately clear that this solution is a limit of some distributional geometry, later on [23] it was shown how to obtain it by a limiting procedure in the  $4D$  gravitating Abelian Higgs model.

In light of this one may argue [24] that to understand the low energy limit of general cosmic strings one ought to look for physically interesting solutions with conical structure in the bulk, even if they include some short distance deformations. We take the view that the example of [22, 23] is a concrete, if fortuitous, evidence in favor of [24], and we follow this directive here. This immediately yields an unexpected prize: the family of exact metrics for a black hole stuck on a codimension-2 brane. This family of solutions is a generalization of the  $4D$  AFV black hole on a string, and includes the very first *explicit*, exact localized

black hole on a 3-brane,<sup>2</sup> that can be used for computing black hole production and decay rates at the LHC [28]. Although we do not have a realization of these solutions as a limit of some distributional source interacting with a black hole, we expect that such a picture should exist, possibly along the lines of the  $4D$  resolution of the AFV solution as in [23].

Our black holes provide us with a direct clue how to find another family of solutions with matter sources localized to a thin brane, where the curvature singularities remain tame even when the matter stress energy is not pure tension. They are exact gravitational fields of a relativistic particle stuck on a thin codimension-2 brane in  $D$  dimensions, and include the fields of photons living on a 3-brane in a  $6D$  flat spacetime. Such solutions can be understood as a brane black hole boosted to a relativistic speed, in a way analogous to the Aichelburg-Sexl solution of GR [29], and just like it carrying only a  $\delta$ -function singularity along its worldline. To obtain the shockwaves, we employ the cutting and pasting techniques of [30, 31] which have already been applied to braneworld models in [32–34], rather than directly boosting the black hole. It turns out that our shockwaves look just like the higher-dimensional shocks [35], which however live on a conical singularity in the bulk, instead of a flat background. Specifically in the case of a 3-brane in  $6D$  they depend on the transverse distance from the source as  $1/r^2$ . To see how to recover  $4D$  long range gravity in this case, we close off the bulk by imposing periodic boundary conditions for bulk fields, as a toy model of compactification. The shockwaves then correctly reproduce the  $4D$  Aichelburg-Sexl solution at distances larger than the period of compactification, whose long-range fields vary as a logarithm of the transverse distance from the source. Examining the black hole and its shockwave limit, we elucidate the short-distance scales at which the nonlinear effects of the gravitational fields of brane-localized objects start to distort the bulk, which should be useful in the search for regulated versions of codimension-2 braneworlds with matter. This supports our view, motivated by [24], that gravity by itself does not really obstruct having localized sources on codimension-2 branes, but may merely obscure the way we see them.

## 2. Field equations and vacua

We start with a brief review of the field equations and vacuum solutions describing tensional straight codimension-2 branes in  $D$  dimensions. We assume that gravity propagates in the bulk as governed by the standard  $D$ -dimensional Einstein-Hilbert action. We further assume that the stress-energy sources are completely localized to a codimension-2 object, vanishing elsewhere in the bulk. This allows us to seek metrics of the form

$$ds^2 = \mathcal{F}^2(y)g_{\mu\nu}(x)dx^\mu dx^\nu + h_{ab}(y)dy^a dy^b, \tag{2.1}$$

where the brane is located at the center of the bulk at  $y^a = 0$ , and is at rest. The field equations in an empty bulk with a brane and a brane-localized stress energy tensor  $T^\mu{}_\nu$

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<sup>2</sup>There exist black hole solutions on a 2-brane in  $4D$  [25] but it is hard to extend them to higher dimensions. Some interpretations of these difficulties were offered in [26], and recently some interesting non-vacuum solutions with  $AdS_4 \subset AdS_5$  asymptotics were studied [27].

are

$$M_D^{D-2} G^A{}_B = T^\mu{}_\nu \delta^A{}_\mu \delta_B{}^\nu \frac{1}{\sqrt{\det h}} \delta^{(2)}(y), \quad (2.2)$$

where the coordinates  $x^\mu$ ,  $\mu \in \{0, \dots, D-3\}$  cover the brane worldvolume and the coordinates  $y^a$ ,  $a \in \{D-2, D-1\}$  parameterize the two dimensions transverse to the brane, while the capital latin indices count over all  $D$  coordinates. With the metrization (2.1), the factor  $1/\sqrt{\det h}$  properly covariantizes the tensor density  $\delta^{(2)}(y)$ . Here  $M_D$  is the bulk Planck mass and  $G^A{}_B$  the bulk Einstein tensor, computed from the full metric (2.1). The induced metric on the brane, from (2.1), is  $\mathcal{F}^2(0)g_{\mu\nu}$ , and as long as  $\mathcal{F}(0)$  is finite we can choose  $\mathcal{F}(0) = 1$  by a rescaling of transverse coordinates  $x^\mu \rightarrow x^\mu/\mathcal{F}(0)$ . Clearly, if  $\mathcal{F}$  diverges as we approach the brane at  $y^a = 0$  things may not be so simple. We will keep this in mind in what follows. Also, in general we could have introduced the cross-terms  $g_{a\mu}$  in the metric (2.1), for example by substituting  $dy^a \rightarrow dy^a + A^a{}_\mu dx^\mu$ . However from this expression it is clear that in the brane worldvolume theory such objects would behave as towers of Kaluza-Klein (KK) vector fields. In what follows we will restrict our attention to the sector where they vanish, assuming that brane sources do not carry KK vector charges. More general solutions with the vectors turned on exist, but are not needed for our purposes here (see [1]).

Tracing (2.2), we obtain

$$M_D^{D-2} R = -\frac{2T}{D-2} \frac{1}{\sqrt{\det h}} \delta^{(2)}(y), \quad (2.3)$$

where  $T = T^\mu{}_\mu$ , and using this we can break up (2.2) into formulas for the transverse and longitudinal Ricci tensor components with respect to the brane worldvolume:

$$M_D^{D-2} R^a{}_b = -\frac{T}{D-2} \delta^a{}_b \frac{1}{\sqrt{\det h}} \delta^{(2)}(y), \quad (2.4)$$

$$M_D^{D-2} R^\mu{}_\nu = \left( T^\mu{}_\nu - \frac{1}{D-2} T \delta^\mu{}_\nu \right) \frac{1}{\sqrt{\det h}} \delta^{(2)}(y), \quad (2.5)$$

alongside the vanishing cross-terms,  $R^a{}_\mu = 0$ . Note that the source on the RHS of the longitudinal equation (2.5) is *traceless*: indeed, if we split the brane stress-energy as the sum of the tension, representing the vacuum energy of the brane matter, and the finite wavelength matter contributions,  $T^\mu{}_\nu = -\lambda \delta^\mu{}_\nu + \tau^\mu{}_\nu$  respectively, we immediately see that the tension, being a part of the trace of  $T$ , immediately cancels from the RHS of (2.5). The longitudinal Ricci tensor components are only sourced by  $T^\mu{}_\nu - T \delta^\mu{}_\nu / (D-2) = \tau^\mu{}_\nu - \tau \delta^\mu{}_\nu / (D-2)$  for any tension  $\lambda$ . On the other hand, the sources for the transverse components of the Ricci tensor  $R^a{}_b$  always depend on  $\lambda$  explicitly. This feature of the field equations (2.2), (2.4), (2.5) is the key ingredient of the magic of ‘off-loading’ brane vacuum energy into the bulk [1, 9]. While it does not guarantee that the induced metric on the brane will always be independent of  $\lambda$ , it does point out that at large distances along the brane the induced metric should be essentially independent of  $\lambda$  when the bulk is asymptotically locally Minkowski.

Interesting solutions of (2.2) which illustrate this desensitization of the induced geometry from  $\lambda$  are easy to find. Suppose that the matter stress-energy vanishes,  $\tau^\mu{}_\nu = 0$ . Since the Ricci tensor is identically zero away from the brane, the field equations (2.2) admit  $D$ -dimensional flat space vacuum with the Minkowski metric as a solution. We factorize the spacetime as a direct product of a  $D - 2$ -dimensional Minkowski and a  $2D$  locally Euclidean and find that the continuity everywhere away from the brane ensures that the warp factor is  $\mathcal{F}^2 = 1$  identically. Thus the metric is exactly

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \delta_{ab}(y) dy^a dy^b, \tag{2.6}$$

where the  $2D$  metric  $\delta_{ab}(y) dy^a dy^b$  is locally Euclidean, but the domain of its definition has to be picked in order to satisfy the equation (2.4) on the brane at  $y^a = 0$  as well as away from it. A complete cover of the transverse space is provided by polar coordinates, in terms of which the metric becomes

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + d\rho^2 + B^2 \rho^2 d\phi^2, \tag{2.7}$$

where  $\phi \in [0, 2\pi]$  and  $B$  is picked to solve (2.4). This yields [36]

$$B = 1 - \frac{\lambda}{2\pi M_D^{D-2}}. \tag{2.8}$$

Thus the longitudinal space covered by the coordinates  $\rho, \phi$  is a cone. It is obtained from the flat disk that would solve (2.2) in the  $\lambda = 0$  limit by extricating a wedge of angular opening  $2\pi(1 - B) = \lambda/M_D^{D-2}$ , identifying the edges of the cut and rescaling the polar angle according to  $\phi \rightarrow B\phi$  [1]. The induced metric on the brane remains flat,  $\propto \eta_{\mu\nu}$ , despite the fact that the brane carried vacuum energy density  $\lambda \neq 0$ .

It should be clear from this that even if the vacuum brane (2.7) is perturbed by a localized matter source described by a  $\tau^\mu{}_\nu \neq 0$  of compact support, the long distance brane geometry may still remain essentially independent of  $\lambda$  as long as the brane straightens out far from the perturbation. Namely, the metric will receive dramatic *gravitational* corrections (at the very least) near the matter source, changing its short distance behavior. Such gravitational short distance corrections should be expected (and were already pointed at in [13, 14]): those effects reflect the nonlinear structure of gravity, accounting for spacetime distortions as do the strong fields near black holes. However if the brane straightens out far from the brane matter perturbations, the bulk geometry far from the brane will converge to the conical Minkowski form where the deficit angle eats up the tension. One may expect that the convergence of the bulk geometry to the vacuum form of (2.7) is rapid, by using the Birkhoff theorem in higher-dimensional gravity and accounting for the deficit angle by appropriately renormalizing Newton's constant. Indeed, if we assume that a regulated perturbed brane exists, then far from the perturbation the field should converge to that of a point mass with a deficit angle. In  $D$  dimensions, the gravitational potential of such an object would fall off as  $1/r^{D-3}$ , where  $r$  is the radial distance away from it, and hence the geometry should rapidly return to that of (2.7). The difficulties with this description should get serious only close in, when nonlinear effects cannot be disregarded. Thus the

scale where the corrections kick up should be on the order of the gravitational radius of the matter perturbation. In the next section, we will confirm this intuition by deriving the exact black hole on a codimension-2 brane, and determining its gravitational radius  $r_0$ .

### 3. Black holes threaded by codimension-2 branes

It is clear from field equations (2.2), (2.4), (2.5) that away from the brane the  $D$ -dimensional Schwarzschild metric,

$$ds_D^2 = -\left(1 - \left(\frac{r_0}{r}\right)^{D-3}\right)dt^2 + \frac{dr^2}{1 - \left(\frac{r_0}{r}\right)^{D-3}} + r^2 d\Omega_{D-2}, \quad (3.1)$$

remains a solution. Here  $r_0$  is the size of the black hole horizon, determined by its mass, and  $d\Omega_{D-2}$  a line element on a unit  $D - 2$  sphere  $S^{D-2}$ . The question is, how is the black hole solution altered in the presence of the brane. In general, even for thin branes whose stress-energy tensor may be imagined to be ultralocal, the presence of the brane may affect dramatically the black hole horizon, and render the explicit determination of the geometry describing a black hole on a brane extremely hard [25, 26].

However, this problem greatly simplifies in the codimension-2 case. To illustrate why, let us first discuss a black hole on a string, given by the AFV solution [22]. Finding solutions of Einstein's equations for a combined gravitational field of some distribution of matter threaded by a string is very easy if the matter distribution has an axial symmetry. In this case, all one needs to do is to orient the string along the axis of symmetry, and account for its presence by cutting a wedge out of the polar variable  $\phi$ , which runs around the symmetry axis. In this way, one obtains the solution whose geometry at infinity approaches the conical space of the string, and close in it gets modified by the gravity of the lump of matter [21, 22]. The AFV black hole is an extreme example of this trick. One simply starts with the  $4D$  Schwarzschild solution, picks the axis, say, in the North-South direction, along the rays  $\theta = 0, \pi$  of the  $S^2$  transverse to the worldline, and replaces the usual  $S^2$  line element by  $d\Omega_2 = d\theta^2 + B^2 \sin^2 \theta d\phi^2$ , choosing  $B$  to still satisfy eq. (2.8) as in the absence of the black hole. Then the Gauss-Bonnet theorem guarantees that the full geometry has the same deficit angle as the string,  $2\pi(1 - B)$ . One can quickly see that this must be the case because far from the hole,  $ds_4^2 \rightarrow -dt^2 + dr^2 + r^2(d\theta^2 + B^2 \sin^2 \theta d\phi^2)$ . Upon substituting  $z = r \cos \theta$ ,  $\rho = r \sin \theta$  this can be rewritten in cylindrical coordinates as  $ds_4^2 \rightarrow -dt^2 + dz^2 + d\rho^2 + B^2 \rho^2 d\phi^2$ , i.e. precisely a locally flat metric with a conical singularity.

We use exactly the same trick to write down the solution describing a black hole on a codimension-2 brane in  $D$  dimensions. This works because, as we have discussed in the previous section, the field of any thin codimension-2 brane in  $D$  dimensions is given by the locally flat metric with a conical singularity. Thus we can just take the higher-dimensional Schwarzschild solution, pick an axis and thread a codimension-2 brane along the axis by cutting out a wedge from the range of the polar angle around this axis, with the opening adjusted to match the tension of the brane according to (2.8).

This 'brane surgery' is most easily performed when we start with the black hole solution in uniform coordinates, in terms of which the metric is of the form  $ds_D^2 = -F dt^2 + G d\vec{x}_{D-1}^2$ . It is straightforward to put the solution (3.1) in this form. We replace the radial variable

$r$  by  $\mathcal{R}$  according to

$$r = \mathcal{R} \left( 1 + \frac{r_0^{D-3}}{4\mathcal{R}^{D-3}} \right)^{\frac{2}{D-3}}, \quad (3.2)$$

which yields

$$ds_D^2 = - \left( \frac{4\mathcal{R}^{D-3} - r_0^{D-3}}{4\mathcal{R}^{D-3} + r_0^{D-3}} \right)^2 dt^2 + \left( 1 + \frac{1}{4} \left( \frac{r_0}{\mathcal{R}} \right)^{D-3} \right)^{\frac{4}{D-3}} \left( d\mathcal{R}^2 + \mathcal{R}^2 d\Omega_{D-2} \right), \quad (3.3)$$

with conformally flat spatial slices. Next we pick a  $D - 3$ -dimensional spatial hypersurface of symmetry (as opposed to merely an axis of symmetry in the  $4D$  AFV case), and transform to cylindrical polar coordinates defined by it, such that  $\vec{x}$  are coordinates along this hypersurface, and we coordinatize the two transverse directions by the transverse distance  $\rho$  and the polar angle  $\phi$ . With these coordinates, we have  $\mathcal{R}^2 = \vec{x}^2 + \rho^2$  and  $d\mathcal{R}^2 + \mathcal{R}^2 d\Omega_{D-2} = d\vec{x}^2 + d\rho^2 + \rho^2 d\phi^2$ . Finally, to thread in a codimension-2 brane with tension  $\lambda$ , we cut a radial wedge in the  $\rho, \phi$  plane of opening  $2\pi(1 - B) = \lambda/M_D^{D-2}$ , according to eq. (2.8), identify the edges, and rescale the angle  $\phi$  to  $\phi \rightarrow B\phi$ , so that after rescaling its range is restored to the interval  $[0, 2\pi)$ . Our final metric is therefore

$$ds_D^2 = - \left( \frac{4(\vec{x}^2 + \rho^2)^{\frac{D-3}{2}} - r_0^{D-3}}{4(\vec{x}^2 + \rho^2)^{\frac{D-3}{2}} + r_0^{D-3}} \right)^2 dt^2 + \left( 1 + \frac{1}{4} \left( \frac{r_0^2}{\vec{x}^2 + \rho^2} \right)^{\frac{D-3}{2}} \right)^{\frac{4}{D-3}} \left( d\vec{x}^2 + d\rho^2 + B^2 \rho^2 d\phi^2 \right), \quad (3.4)$$

and it represents a black hole, of horizon size  $r_0$ , stuck on a codimension-2 brane. In fact, we should note that it is straightforward to go back to the spherical polar coordinates for the metric (3.4) with the brane included. All we would do is basically return to the Schwarzschild metric (3.1), but with the line element  $d\Omega_{D-2}$  on the unit sphere  $S^{D-2}$  replaced by the line element  $d\ell_{D-2}^2 = d\Omega_{D-3} + B^2 \prod_{k=1}^{D-3} \sin^2(\theta_k) d\phi^2$ , which is the metric on a unit  $D - 2$ -dimensional sphere but with a wedge of opening  $2\pi(1 - B)$  removed from the polar angle  $\phi$ . This means that the spatial surfaces of constant radius are topologically spheres, pinched on the brane by the tension-induced deficit angle. We should also note that among the black hole solutions (3.4) probably the most phenomenologically interesting one is  $D = 6$ , where our solution models an exact small  $6D$  black hole residing on a 3-brane in two extra dimensions,

$$ds_6^2 = - \left( \frac{4(\vec{x}^2 + \rho^2)^{3/2} - r_0^3}{4(\vec{x}^2 + \rho^2)^{3/2} + r_0^3} \right)^2 dt^2 + \left( 1 + \frac{1}{4} \left( \frac{r_0^2}{\vec{x}^2 + \rho^2} \right)^{3/2} \right)^{4/3} \left( d\vec{x}^2 + d\rho^2 + B^2 \rho^2 d\phi^2 \right), \quad (3.5)$$

which can be used for precise and explicit calculations of production and evaporation of quantum black holes at the LHC, as in the studies of [28].

Let us (very!) briefly review some of the properties of the black hole family (3.4). As in the case of the AFV solution [22], the horizon distance  $r_0$  is an integration constant in (3.4), and as such independent of brane tension. So for a fixed  $r_0$  the surface gravity and the Hawking temperature of the hole are completely independent of the brane. The Euclideanized version of the solution (3.1) then readily yields that the Hawking temperature, defined by the period of the Euclidean time, is  $T_H = \frac{D-3}{4\pi r_0}$ . However, the presence of the brane alters the relation between the horizon size and the mass of



the black hole controlling its inertia, as measured by the hole's momentum integrals at asymptotic infinity. More formally, the formula for the ADM mass of the black hole is corrected because of the deficit angle. To see this, we can look at the linearized form of the hole metric (3.4), which, using spherical polar coordinates,  $d\Omega_{D-2} \rightarrow d\ell_{D-2}^2$ , is  $ds_D^2 = -\left(1 - (r_0/\mathcal{R})^{D-3}\right)dt^2 + \left(1 + \frac{1}{D-3}(r_0/\mathcal{R})^{D-3}\right)(d\mathcal{R}^2 + \mathcal{R}^2 d\ell_{D-2}^2)$ . So the mass of the hole is [37]

$$m = \frac{D-2}{2} M_D^{D-2} r_0^{D-3} \int_{\text{angles}} d\ell_{D-2}. \tag{3.6}$$

Since angles run over a  $D-2$ -dimensional sphere with a deficit angle, the integral is given by  $\Omega_{D-2} B$ , where  $\Omega_{D-2} = 2\pi^{\frac{D-2}{2}}/\Gamma(\frac{D-2}{2})$  is the volume of a unit  $S^{D-2}$  and  $B$  is the deficit angle parameter in (2.8). Introducing a shorthand  $\alpha_D = (D-2)\Omega_{D-2}/2$  for the fixed dimensionless quantities, the mass is<sup>3</sup>

$$m = \alpha_D M_D^{D-2} B r_0^{D-3}. \tag{3.7}$$

Inverting, we find that the horizon size  $r_0$  is expressed in terms of the ADM mass  $m$  according to  $r_0 = m^{1/(D-3)}/(\alpha_D M_D^{D-2} B)^{1/(D-3)}$ , or, using (2.8),

$$r_0 = \left(\frac{2\pi}{2\pi M_D^{D-2} - \lambda}\right)^{\frac{1}{D-3}} \left(\frac{m}{\alpha_D}\right)^{\frac{1}{D-3}}. \tag{3.8}$$

Now, it is clear from the black hole solution (3.4) and its linearized form that the strong gravity effects and nonlinear corrections begin to affect the geometry at distances of the order of  $r_0$  from the hole. Because of the equivalence principle, however, this will remain true even for sources which have not yet collapsed, but may be stabilized by some matter interactions. From formula (3.8) it is clear that the actual scale where this happens depends not only on the mass sourcing the field, but also on the tension of the brane. For a fixed value of mass, nonlinear gravity effects could start at distances much greater than a naive estimate of the gravitational radius based on a 'braneless' higher dimensional gravity,  $\propto M_D^{-1}(m/M_D)^{1/(D-3)}$ , because of the conical enhancement of the gravitational force, as is manifest in (3.8). The closer the tension is to the bulk scale, which would be expected by naturalness, and needed to avoid a large  $4D$  vacuum energy upon compactification [38], the larger the gravitational radius of the mass  $m$ ! In effect, the codimension-2 brane behaves as a lightning rod for gravity: the deficit angle lessens the bulk volume near the brane, which hampers dilution of gravitational force with distance. Note however that we confirm the intuition that a version of the higher-dimensional Birkhoff theorem still applies for masses on a codimension-2 brane, despite the presence of the brane. The potential drops off as claimed, according to  $1/r^{D-3}$ , although the scale beyond which the nonlinear effects are negligible may be pushed out to distances  $\gg M_D^{-1}(m/M_D)^{1/(D-3)}$ . In practice, this implies that in the attempt to regulate the brane in order to deal with the effects of strong

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<sup>3</sup>Note that the Bekenstein-Hawking entropy  $\sim$  area law for black holes,  $S \sim A/G_N$ , is properly upheld. Plugging in this equation the area,  $A \sim r_0^{D-2}$ , and the coupling on the cone,  $G_N \sim 1/(M_D^{D-2} B)$ , we find that  $S \sim (r_0 M_D)^{D-2} B$ , and so  $T_H S \sim S/r_0 \sim r_0^{D-3} M_D^{D-2} B$ , or therefore  $T_H S \sim m$  (using eq. (3.7)).

gravity of some object of mass  $m$  as in [13]–[18], one must thicken up the brane to exceed the gravitational radius of the mass  $m$ , as given by (3.8), in order to be able to treat gravity perturbatively, and depending on the brane tension this scale could be very large. Gravitational shockwaves, which we turn to next, provide us with further examples of this gravitational lightning rod phenomenon.

#### 4. Gravitational shockwaves

As we have seen above, the nonlinear gravitational corrections at short distances cannot be neglected any more at scales comparable to the gravitational radius  $r_0$  of the source. Although this distance may depend on the mass in a complicated way (3.8) because of the environmental effects, it really comes about because the mass of the source breaks the conformal symmetry of the background. Clearly, the smaller the mass, the shorter the scale where gravitational nonlinearities become large. This immediately points how to regain some level of mathematical control over the nonlinearities in the theory, while continuing to explore nontrivial sectors of gravity. The trick is to try to suppress the scale at which conformal symmetry is broken, while keeping a nontrivial stress-energy source to generate gravity. Clearly, restoring conformal symmetry means looking at sources whose stress-energy has negligible or vanishing trace. Hence we should look at the gravitational fields of very fast particles on the brane. Their gravitational field will be sourced by the momentum, and the distance scale below which the nonlinearities are significant will be arbitrarily short, controlled by the ratio of the rest mass to the momentum of the particle. In the ultrarelativistic limit, when the rest mass vanishes, we would expect that the linearized gravity description would remain valid down to extremely short distances, in which case we should be able to retain the thin-brane description of relativistic stress-energy as a  $\delta$ -function source. Indeed, this is precisely how the gravitational shockwave solutions work in conventional GR and in the theories with branes [29]–[34]. The relativistic limit suppresses the scale where nonlinearities kick in by restoring the conformal symmetry of the matter sector, which in turn allows a linear description all the way to arbitrarily short distances.

To confirm this intuition, we construct the explicit form of the gravitational shockwaves, sourced by relativistic particles, such as a photon, on a codimension-2 brane. To do so, we could have followed the road Aichelburg and Sexl set out on in their seminal paper [29]: take our black hole (3.4), linearize it, and boost it until its worldline becomes null (but ensure that in this process we properly gauge-fix the linearized solution so that no non-physical divergences are encountered [29]). However, a simpler method is to note that because of the Lorentz contraction generated by the boosting, the gravitational field of the relativistic particle will be completely confined to the transverse plane, orthogonal to the instantaneous location of the particle. Hence, before and after that surface, the space will be vacuum, and the only nontrivial information about the field will be contained in the junction conditions at this surface, which separates these vacuum regions. This enables us to use the cutting and pasting technique of [30, 31], which has already been successfully used in braneworld models [32–34].

So, as in those cases, we start with the vacuum codimension-2 brane solution (2.7), pick a direction on the brane and switch to lightcone coordinates along it. To encode the shock wave, we put a relativistic particle along one of the null lines, say  $u = 0$ , and introduce a discontinuity in the orthogonal null coordinate  $v$  by replacing  $dv$  in the metric by  $dv - f(\vec{x}_\perp, \rho, \phi)\delta(u)du$  [30]–[34]. Here  $\vec{x}_\perp$  denotes the spatial dimensions along the brane which are orthogonal to the direction of motion of the relativistic source. The shocked metric then becomes

$$ds_D^2 = 4dudv - 4\delta(u)fdu^2 + d\vec{x}_\perp^2 + d\rho^2 + B^2\rho^2d\phi^2. \quad (4.1)$$

Here  $f(\vec{x}_\perp, \rho, \phi)$  is the shockwave profile, which only depends on the spatial directions transverse to the motion,  $\vec{x}_\perp$  along the brane and  $\rho, \phi$  away from it. Further, we add to the brane stress-energy tensor  $T^\mu{}_\nu$  the contribution from the momentum of the relativistic particle, given in terms of the shocked induced brane metric  $g_{D-2\mu\nu}$  in (4.1) by [30]–[34]

$$\tau^\mu{}_\nu = \frac{2p}{\sqrt{g_{D-2}}} g_{D-2uv} \delta(u) \delta^{(D-4)}(\vec{x}_\perp) \delta^\mu{}_\nu \delta^u{}_\nu. \quad (4.2)$$

What remains is to substitute (4.1) and (4.2) into the field equations (2.4), (2.5) and work out the field equation for the shockwave profile  $f$ . Because  $\tau^\mu{}_\nu$  is traceless, it does not enter in the transverse field equations (2.4), which therefore remain identical to the vacuum case, and are solved automatically by (4.1) provided that (2.8) holds. On the other hand, because the tension term cancels in the longitudinal equations (2.5), as discussed in the text following eq. (2.5), and  $\tau^\mu{}_\mu = 0$ , we find

$$R^\mu{}_\nu = \frac{2p}{M_D^{D-2} B \rho} \delta(u) \delta^{(D-4)}(\vec{x}_\perp) \delta(\rho) \delta(\phi) \delta^\mu{}_\nu \delta^u{}_\nu, \quad (4.3)$$

where we have used  $g_{D-2uv}/\sqrt{g_{D-2}} = 1$ , and  $h_{ab} = \text{diag}(1, B^2\rho^2)$  for the metric transverse to the brane, as per (4.1). The only component of  $R^\mu{}_\nu$  which does not vanish trivially is  $R^v{}_u$ , and its direct evaluation along the lines of, for example [33, 34], yields

$$R^v{}_u = \delta(u) \nabla_{D-2}^2 f, \quad (4.4)$$

where  $\nabla_{D-2}^2 f = \vec{\nabla}_\perp^2 f + \Delta_2 f$  is the Laplacian defined with respect to the part of the metric (4.1) transverse to the shockwave, spanned by the coordinates  $\vec{x}_\perp$  and  $\rho, \phi$ , respectively. Comparing (4.3) and (4.4) yields the equation for the shockwave profile that we were after:

$$\nabla_{D-2}^2 f = \frac{2p}{M_D^{D-2} B \rho} \delta^{(D-4)}(\vec{x}_\perp) \delta(\rho) \delta(\phi). \quad (4.5)$$

This is the equation for the static potential of a ‘charge’  $p$  at the origin, on the tip of the cone in  $D - 2$ -dimensional space, which generates a force with a coupling strength  $g \sim \frac{1}{M_D^{D-2} B}$ . It is straightforward to write its solution, which is<sup>4</sup>

$$f = -\frac{1}{(D-4)\Omega_{D-3}} \frac{2p}{M_D^{D-2} B} \frac{1}{(\vec{x}_\perp^2 + \rho^2)^{\frac{D-4}{2}}}, \quad (4.6)$$

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<sup>4</sup>Since the Laplacian is  $D - 2$ -dimensional, and the ‘charge’ is at the origin, the solution must be of the form  $f = \frac{\mathcal{Q}}{R^{D-4}}$ , where  $R^2 = \vec{x}_\perp^2 + \rho^2$ . The normalization can be determined from applying Gauss law to (4.5), yielding  $\int d\vec{S} \cdot \vec{\nabla} f = \frac{2p}{M_D^{D-2} B}$  and so  $\mathcal{Q} = -\frac{2p}{(D-4)\Omega_{D-3} M_D^{D-2} B}$ .

where  $\Omega_{D-3} = \frac{2\pi^{\frac{D-2}{2}}}{\Gamma(\frac{D-2}{2})}$  is the volume of a unit  $S^{D-3}$ . Note that the gravitational lightning rod effect, which we observed in the previous section, remains manifest in (4.5). Due to the conical background, the effective coupling is renormalized from  $1/M_D^{D-2}$  to  $1/(M_D^{D-2}B)$ , and so it is sensitive to the brane tension:  $g \sim \frac{2\pi}{2\pi M_D^{D-2-\lambda}}$ . Thus the gravitational coupling becomes very strong as the tension approaches the fundamental scale  $M_D$ . However, the gravitational nonlinearities remain under control, being completely suppressed in the relativistic limit by the boosting of the source. We remark that the solution (4.6) is so simple despite the conical structure of space because the stress-energy source is on the brane, or equivalently the effective ‘charge’ is on the tip of the transverse cone. For a source in the bulk off the tip, the potential of (4.6) would be more complicated. At distances short compared to the displacement of the ‘charge’ from the tip the potential would be the same as in a flat bulk, without coupling enhancement as the tip is too far to affect it. It would asymptotically approach (4.6) as distance increases [39], and would reduce exactly to it as the ‘charge’ is moved back to the tip of the cone. At any rate, the solution (4.6) encapsulates the correct long distance behavior of the shockwave. We can finally write down the gravitational field of a relativistic particle zipping along a codimension-2 brane in  $D$ -dimensional space time:

$$ds_D^2 = 4dudv + \frac{8p}{(D-4)\Omega_{D-3}M_D^{D-2}B} \frac{\delta(u) du^2}{(\vec{x}_\perp^2 + \rho^2)^{\frac{D-4}{2}}} + d\vec{x}_\perp^2 + d\rho^2 + B^2\rho^2 d\phi^2. \quad (4.7)$$

In fact this solution looks the same as the higher-dimensional shockwave in a locally flat spacetime [35], the only exception being the conical enhancement of the coupling. The solution (4.7) is an exact solution of the field equations (2.2), (2.4), (2.5), the brane is thin, with a  $\delta$ -function tension as in the vacuum case, but the total stress-energy tensor on the brane is manifestly *not* equal to pure tension, as can be seen from (see eqs. (2.2), (4.2))

$$T^A_B = \left( -\lambda\delta^\mu_\nu + \frac{2p}{\sqrt{g_{D-2}}} g_{D-2uv} \delta(u) \delta^{(D-4)}(\vec{x}_\perp) \delta^\mu_\nu \delta^u_\nu \right) \delta^A_\mu \delta_B^\nu \frac{1}{\sqrt{\det h}} \delta^{(2)}(y). \quad (4.8)$$

The solution (4.7) remains under control down to extremely short distances. The reason the shockwave (4.7) evades the results of [13, 14] is that in the relativistic limit the gravitational nonlinearities remain completely under control, as we have discussed above. In (4.7), (4.8) it is clear where the nonlinearities have ‘gone’: they have been pushed into the metric discontinuity  $\propto \delta(u)$  along the worldline of the source in (4.7), (4.8). While this  $\delta$ -function may appear frightful at the first glance, in fact its divergence is a coordinate artifact that can be easily removed by a diffeomorphism discussed in [40, 41]. Using (4.1) and (4.6) for notational brevity, note first that the shockwave is axially symmetric,  $\partial_\phi f = 0$ . Further introduce new notation, defining  $X^i = (\vec{x}_\perp, \rho)$ , such that (4.1) becomes  $ds_D^2 = 4dudv - 4\delta(u)fdv^2 + \delta_{ij}dX^i dX^j + B^2\rho^2 d\phi^2$ . Then define new coordinates  $u = \tilde{u}$ ,  $v = \tilde{v} + f\theta(\tilde{u}) - \tilde{u}(\nabla f)^2$ ,  $X^i = \tilde{X}^i - 2\tilde{u}\theta(\tilde{u})\tilde{\partial}_i f$ , where  $\theta(\tilde{u})$  is the step function, and new variables are substituted in place of the old ones in the function  $f$  in these transformations. Using  $d[\theta(\tilde{u})] = \delta(\tilde{u})d\tilde{u}$  and  $\tilde{u}\delta(\tilde{u}) \equiv 0$ , and noting that  $\delta(u)f(X) = \delta(\tilde{u})f(\tilde{X})$ , we can substitute this change of variables in the metric (4.1), (4.6) to get, after a straightforward but tedious

calculation, the expression

$$ds_D^2 = 4d\tilde{u}d\tilde{v} + \left( \delta_{ij} - 4\tilde{u}\theta(\tilde{u})\tilde{\partial}_i\tilde{\partial}_j f + 4\tilde{u}^2\tilde{\partial}_i\tilde{\partial}_k f\tilde{\partial}_j\tilde{\partial}^k f \right) d\tilde{X}^i d\tilde{X}^j + B^2(\tilde{\rho} - 2\tilde{u}\theta(\tilde{u})\tilde{\partial}_\rho f)^2 d\phi^2, \tag{4.9}$$

with the form of  $f(\tilde{X})$  given in (4.6). This metric is manifestly well-behaved at  $\tilde{u} = 0$ .

There is still the singularity at the core of the source, at  $\vec{x}_\perp = \rho = 0$ . Clearly, at any finite distance  $|\vec{x}_\perp| > 0$  from the source along the brane, there is no bulk divergence at all. The only singular limit arises in the case of first approaching  $\vec{x}_\perp = 0$  away from the brane, and then moving up to it, at the tip of the cone. Although this singularity does not infect the Ricci curvature, it will show up in the Riemann tensor, that depends on objects like  $\partial_j\partial_k f$ . This however is the usual short distance singularity associated with any potential source, familiar from electrostatics or Newtonian gravity. In any case, one expects that at some very short distance this singularity can be consistently smoothed out by matter sector effects alone, for example by quantum mechanical fuzzing up of the source. Therefore, the solution (4.7) is under control as a representation of the gravitational field of a brane-localized particle. This shows that brane-localized sources by themselves are not the culprit of the difficulties with matter-laden thin branes encountered in [13, 14], and subsequently investigated in [15]–[18]. The real cause of these problems is that gravity is not  $4D$  close in, and so it spreads into the bulk causing strong nonlinear deformations at distances on the order of the gravitational radius of the energy lump. But this should be expected all along.

### 5. $4D$ limits

Having realized what the subtleties with placing matter sources on thin codimension-2 branes are, it is natural to ask once matter is included how one can recover  $4D$  gravitational force at large distances. Using a modification of our shockwave geometry (4.7), we will argue here that the recovery of  $4D$  Newton’s law may proceed as usual once the scales in the theory are properly accounted for. We will focus on the case of a tensional 3-brane in a  $6D$  spacetime, although extending the argument to more dimensions with wrapped branes should be straightforward. To proceed, let us close the bulk off in some way at a finite distance from the 3-brane. This could be done in various ways (see [1]–[6], [38] for examples). The simplest approach to recovering  $4D$  gravity however is to ignore all the details of compactification, and merely ask if the correct law at large distances can be so retrieved.

A simple way to check if this happens is to model the compactification by imposing some boundary conditions in the bulk, that remove the ‘exterior’. A natural trick would be to use Neumann boundary conditions because they force the gradient of the potential which would solve (4.5) to vanish on some boundary in the radial direction from our 3-brane. This means that the radial component of the field strength would vanish, and that the field lines bend around to become parallel with the 3-brane, so that they stop diluting in the transverse directions. Thus the field strength must switch to the  $4D$  law at

large distances. However, implementing this procedure directly on a stationary field in a compact space requires introducing unphysical ‘negative energy’ sources on the boundary, by Gauss’s law, and dealing with them, while possible, is unwieldy [42].

To circumvent these issues, we will instead use periodic boundary conditions, imposing them by placing images of the brane throughout the infinite bulk. Although the 3-brane is a cone in the transverse dimensions, and it is hard to picture a periodic array of such cones, we will use the fact that the deficit angle factors into the enhanced gravitational coupling as in (4.6), and treat (4.6) as the shockwave on a plane. This should be sufficient for our purposes here. Clearly, a consistent compactification mechanism would have to be devised to properly account for such short distance issues, but we can nevertheless test in this way if it can be expected to reproduce 4D gravity at all. So let us imagine that (4.6) is promoted into a 2D lattice by translations in the two bulk directions along orthogonal unit vectors  $\vec{e}_1$  and  $\vec{e}_2$ . By linear superposition, the total shockwave profile of such an array in  $D = 6$  will be

$$f_{compact} = -\frac{p}{2\pi^2 M_6^4 B} \sum_{n_1, n_2 = -\infty}^{\infty} \frac{1}{\vec{x}_\perp^2 + (\vec{\rho} - n_1 L \vec{e}_1 - n_2 L \vec{e}_2)^2}, \quad (5.1)$$

where  $\vec{\rho}$  is the bulk component of the radius vector from the 3-brane at the origin to the point where the potential is measured, and  $L$  is the lattice spacing. We can restrict to  $|\vec{\rho}| \lesssim L$ . At large distances on the brane transverse to the shock source,  $|\vec{x}_\perp|^2 \gg L^2$ , we can approximate the sum by an integral. Replacing  $n_{1,2} \rightarrow y_{1,2}$  (with this normalization  $y_k$  are dimensionless), we note that

$$\sum_{n_1, n_2 = -\infty}^{\infty} \frac{1}{\vec{x}_\perp^2 + (\vec{\rho} - n_1 L \vec{e}_1 - n_2 L \vec{e}_2)^2} \rightarrow \frac{1}{L^2} \int_{\text{plane}} \frac{d^2 \vec{y}}{(\vec{y} - \vec{\rho}/L)^2 + \vec{x}_\perp^2/L^2}. \quad (5.2)$$

To evaluate the integral (5.2) over an infinite plane, we shift the origin by a bulk translation  $\vec{y} \rightarrow \vec{y} + \vec{\rho}/L$ , without changing the measure of integration, and then using axial symmetry around the center brane integrate over the polar angle  $\phi$  about it. This yields

$$\frac{1}{L^2} \int_{\text{plane}} \frac{d^2 \vec{y}}{\vec{y}^2 + \vec{x}_\perp^2/L^2} = \frac{2\pi}{L^2} \int_0^\infty dy \frac{y}{y^2 + \vec{x}_\perp^2/L^2}, \quad (5.3)$$

The remaining integral is formally infinite because of the logarithmically divergent contribution of the upper limit of integration. This infinity is an *unphysical* infra-red divergence arising from the contributions of ‘charges’ infinitely far away, because their uniform number density far away overcompensates the potential shutdown with distance. The infinity is unphysical since the divergent term is pure gauge, and we can remove it with a diffeomorphism. To do so, we should first regulate the integral (5.3) with a coordinate space cutoff  $\Lambda \gg |\vec{x}_\perp|/L$ , which yields

$$\frac{2\pi}{L^2} \int_0^\infty dy \frac{y}{y^2 + \vec{x}_\perp^2/L^2} \rightarrow \frac{2\pi}{L^2} \int_0^\Lambda dy \frac{y}{y^2 + \vec{x}_\perp^2/L^2} = \frac{\pi}{L^2} \ln\left(\frac{\Lambda^2 + \vec{x}_\perp^2/L^2}{\vec{x}_\perp^2/L^2}\right). \quad (5.4)$$

Next we decompose the logarithm as

$$\begin{aligned} \frac{\pi}{L^2} \ln\left(\frac{\Lambda^2 + \vec{x}_\perp^2/L^2}{\vec{x}_\perp^2/L^2}\right) &= \frac{2\pi}{L^2} \ln \Lambda - \frac{2\pi}{L^2} \ln\left(\frac{|\vec{x}_\perp|}{L}\right) + \frac{\pi}{L^2} \ln\left(1 + \frac{\vec{x}_\perp^2}{\Lambda^2 L^2}\right) \\ &= \frac{2\pi}{L^2} \ln \Lambda - \frac{2\pi}{L^2} \ln\left(\frac{|\vec{x}_\perp|}{L}\right) + \frac{\pi}{L^2} \frac{\vec{x}_\perp^2}{\Lambda^2 L^2} + \dots \end{aligned} \quad (5.5)$$

where we have expanded the last logarithm in the top line using  $\Lambda \gg |\vec{x}_\perp|/L$ . Further, we substitute (5.5) into (5.1), and simultaneously perform the coordinate transformation

$$v \rightarrow v + \mathcal{A}\theta(u), \quad (5.6)$$

in the metric (4.1), where  $\mathcal{A}$  is a constant yet to be determined and  $\theta(u)$  the step function. Under this transformation, the shockwave profile changes to

$$f \rightarrow f - \mathcal{A}. \quad (5.7)$$

Then we set  $\mathcal{A} = -\frac{p}{\pi L^2 M_6^4 B} \ln \Lambda$ . This completely cancels the divergent term in the transformed  $f_{compact}$ , allowing us to take the limit  $\Lambda \rightarrow \infty$  at will. In this limit, all the cutoff-dependent polynomial corrections  $\propto 1/\Lambda^{2n}$  in (5.5) vanish without a trace. Hence as we promised, the divergence is completely gauged away, leaving no effect behind. After introducing the 4D Planck mass  $M_4^2 = L^2 M_6^4 B$ , which is precisely the correct Gauss law formula including the area of the extra-dimensional space, restricted to an elementary cell of the lattice, we finally find that at large distances along the brane the shockwave converges to

$$f_{compact} = \frac{p}{\pi M_4^2} \ln\left(\frac{|\vec{x}_\perp|}{L}\right), \quad (5.8)$$

The shockwave profile of eq. (5.8) is *precisely* the Aichelburg-Sexl 4D shockwave solution correctly weighed with the 4D Planck's constant — just as we have claimed! We see that the compactification by periodic boundary conditions has reproduced the 4D limit of the solution, with the correctly normalized 4D Planck mass, including the enhancement by the deficit angle. The exact matching of the numerical coefficients should not be surprising in spite of the simplicity of the setup, because of its covariance. Our ‘compactification prescription’ merely introduced image ‘charges’ which restrict the bulk space to a finite volume without disturbing the setup. The resulting periodicity together with the positivity of the potential imply that there must exist equipotential surfaces around each charge in the lattice where the potential takes its minimum, and so has vanishing gradients. This construction is thus effectively imposing Neumann boundary conditions on the potential minimal surfaces, without any auxiliary negative ‘charges’. Based on our results, we expect that detailed compactification mechanisms with general matter on codimension-2 branes should work out when the matter disturbances of the compactification dynamics and the proper regulators of the matter-laden 3-brane are determined using all the relevant scales in the problem.

## 6. Summary

In this note we have derived exact black hole and shockwave solutions localized on a codimension-2 brane in  $D$  dimensions, with gravity governed by the bulk  $D$ -dimensional Einstein-Hilbert action. The black hole solutions are higher-dimensional Schwarzschild geometries with a polar deficit angle, which is interpreted as a manifestation of the brane's tension that renders the bulk conical. The solutions are a generalization of the AFV black hole pierced by a cosmic string in  $4D$  [22]. They include a  $6D$  black hole on a 3-brane, which can be viewed as an explicit example of a 'small' black hole residing on a 3-brane in theories with large extra dimensions, with the horizon size smaller than the size of the extra dimensions, which should be an interesting arena for explicit calculations of black hole production and decay rates at the LHC [28]. Note, that although our solution (3.5) reduces to  $6D$  Schwarzschild when the brane tension is much smaller than the fundamental scale, when the tension is large a black hole with a fixed mass, given by the Center-of-Mass energy of the collision in which it is created, should have a larger radius as dictated by eq. (3.8), and hence a greater entropy. This may improve the semiclassical approximation used to compute black hole evolution. It would be interesting to test the precise prediction with the brane tension contributions included, and also seek out other black hole examples, e.g. with charges and angular momenta.

Our shockwave solutions can be viewed as infinite boost limits of brane-localized black holes, although we find them by employing the cut-and-paste tricks of Dray and 't Hooft [30]. They provide an explicit demonstration that gravity really does not obstruct having localized sources on codimension-2 branes, but merely obscures their mathematical description because of the strong nonlinearities at distances comparable to the gravitational length of the source. For relativistic particles, the boost restores scaling symmetry pushing the gravitational radius to zero, and putting nonlinear effects under control. Thus relativistic particles can be easily described as matter sources on thin branes, with  $\delta$ -function stress-energy. The residual short distance singularities that appear as the distance from the source goes to zero should be expected to be resolved as usual, by short distance physics in the core of the source, as for example the Coulomb singularities of electrostatics which get smeared by quantum effects. In the case of an infinite locally flat bulk, the shockwave profiles drop off with distance as  $1/r^{D-4}$ , i.e. as  $1/r^2$  on a 3-brane in  $6D$ , manifestly displaying the dimensionality of the full spacetime. As a check, we reconsider the shockwave on a 3-brane when we close the bulk off by imposing periodic boundary conditions with a lattice spacing  $L$ . In this case we recover the correct logarithmic variation with distance of the  $4D$  Aichelburg-Sexl shockwave at transverse distances along the brane larger than  $L$ . These examples support our view that there exist solutions sourced by stress-energy other than tension, independently of the internal structure of the brane and without ever putting higher derivative graviton operators in the bulk. In general, however, to regulate their mathematical description correctly, in order to restore the thin brane limit at large distances, one must account properly for the scales where the nonlinearities of the gravitational field become important. While that may be technically involved, it should be possible in principle.



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*Alternatively, one can adopt a more cavalier approach and simply look for solutions of Einstein's equations with conical singularities, leaving the mathematical purist to resolve the question of whether or not they can be interpreted as arising from distributional sources...*
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therefore  $(1 - B)\vec{\nabla}_z^2 \ln(|\vec{z}|/\ell) = \frac{\lambda}{M_D^{D-2}}\delta^{(2)}(\vec{z})$ . But because  $\ln(|\vec{z}|/\ell)$  is the Euclidean  $2D$  Green's function,  $\vec{\nabla}_z^2 \ln(|\vec{z}|/\ell) = 2\pi\delta^{(2)}(\vec{z})$ , so by comparison  $2\pi(1 - B) = \lambda/M_D^{D-2}$ .

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